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INTERPOLATION IN THE UNIT BALL OF $\mathbf{C}^n$

BY

BAO QIN LI* AND ENRIQUE VILLAMOR

Department of Mathematics, Florida International University
University Park, Miami, FL 33199, USA
e-mail: libaoqin@fiu.edu and Villamor@fiu.edu

ABSTRACT

A necessary and sufficient condition is given for a discrete multiplicity variety in the unit ball $\mathbf{B}_n$ of $\mathbf{C}^n$ to be an interpolating variety for weighted spaces of holomorphic functions in $\mathbf{B}_n$.

1. Introduction

In this paper, we shall consider the interpolation problem for discrete multiplicity varieties in the unit ball $\mathbf{B}_n$ of $\mathbf{C}^n$ to be interpolating varieties for holomorphic functions with growth conditions.

Let f be a holomorphic function in $\mathbf{B}_n$ and $\{\zeta_k\}$ a discrete set in $\mathbf{B}_n$. Then we have the following Taylor expansion about each ζ_k:

$$f(z) = \sum_{|I|=0}^{\infty} f_{k,I}(z-\zeta_k)^I.$$

Here and throughout the paper

$$f_{k,I} = \frac{1}{I!}\frac{\partial^{|I|} f(\zeta_k)}{\partial z^I},$$

$I := (i_1, \ldots, i_n) \in (\mathbf{Z}^+)^n$ is a multi-index, $\mathbf{Z}^+ = \{0, 1, 2, \ldots\}$, and

$$|I| = i_1 + i_2 + \cdots + i_n.$$

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Let $\{m_k\}$ be a sequence of positive integers. If for any multi-indexed sequence $\{a_{k,I}\}_{k\in\mathbf{N},0\le|I|<m_k}$ of complex numbers satisfying a certain growth condition (defined in §2) there exists a holomorphic function f in $\mathbf{A}^{-\infty}(\mathbf{B}_n)$ such that

$$f_{k,I} = a_{k,I}, \quad \text{for } k \in \mathbf{N}, \quad 0 \le |I| < m_k, \tag{1.1}$$

where $I = (i_1, \dots, i_n) \in (\mathbf{Z}^+)^n$ is a multi-index, and $\mathbf{A}^{-\infty}(\mathbf{B}_n)$ is the sapce of holomorphic functions in $\mathbf{B}_n$ satisfing

$$(1 - |z|)^A |f(z)| < B, \quad z \in \mathbf{B}_n$$

for some constants $A, B > 0$, we will then say that $V := \{(\zeta_k, m_k)\}$ is an interpolating (multiplicity) variety for $\mathbf{A}^{-\infty}(\mathbf{B}_n)$. Note that the condition (1.1) means that f has a prescribed finite collection of Taylor coefficients at each ζ_k. In the special case that $m_k = 1$ for all k, (1.1) simply means that f takes prescribed values at each ζ_k.

Similar interpolation problems for weighted spaces of entire functions in $\mathbf{C}^n$ have been studied extensively due to its applications to other subjects such as harmonic analysis and systems theory (see [BG2], [BL1], [BL2], [BT], [LV], [S], etc.). Given a discrete set $V = \{\zeta_k\}$ in $\mathbf{C}^n$, a necessary and sufficient interpolation condition in terms of the Jacobian of defining functions was found in [BL1] for V to be an interpolating variety for the space $A_p(\mathbf{C}^n)$, the Hörmander algebra of entire functions in $\mathbf{C}^n$ satisfying $\sup_{z\in\mathbf{C}^n}\{e^{-Ap(z)}|f(z)|\} < +\infty$ for some $A > 0$ in the sense of Berenstein and Taylor ([BT]), where p is a plurisubharmonic weight function in $\mathbf{C}^n$. It was showed in [M] that this condition can be carried over to $\mathbf{B}_n$ for a discrete set $\{\zeta_k\}$ in $\mathbf{B}_n$ to be an interpolating variety for $\mathbf{A}^{-\infty}(\mathbf{B}_n)$, a class intensively studied when $n = 1$ (see e.g. [K]). Since in practice the "multiplicity" often needs to be taken into account (cf. [BT]), it is natural to study the interpolation problem and interpolation conditions for multiplicity varieties $V = \{(\zeta_k, m_k)\}$ in the unit ball $\mathbf{B}_n$. When no multiplicity is involved, the notion of interpolation variety in $\mathbf{B}_n$ for $\mathbf{A}^{-\infty}(\mathbf{B}_n)$ is the same as the one in $\mathbf{C}^n$ for $A_p(\mathbf{C}^n)$ with an obvious change in weight and domain. However, when one consideres multiplicity problems, the situation will be different. The differences will be pointed out in Section 2 and the interpolation problem for multiplicity varieties will then be posed, which natually includes the usual problem without multiplicity as a special case. A necessary and sufficient interpolation condition will be given for a multiplicity variety $V = \{(\zeta_k, m_k)\}$ to be an interpolating variety for $\mathbf{A}^{-\infty}(\mathbf{B}_n)$. The conditions obtained here, inspired by the approachs in [BT] and [LV] for interpolation in $\mathbf{C}^n$, are given using the distribution of points of V in the "tube"

$$(1.2) \qquad S(F;\epsilon,C) := \{z \in \mathbf{B}_n : |F(z)| := (\sum_{j=1}^{n} |f_j(z)|^2)^{\frac{1}{2}} < \epsilon(1-|z|)^C\},$$

where $F = (f_1, \ldots, f_n)$ is a defining vector function with $f_j \in \mathbf{A}^{-\infty}(\mathbf{B}_n)$ and $\epsilon, C > 0$ are two constants. It turns out that a multiplicity variety $V = \{(\zeta_k, m_k)\}$ in $\mathbf{B}_n$ is an interpolating variety for $\mathbf{A}^{-\infty}(\mathbf{B}_n)$ if and only if there exist constants $\epsilon, C > 0$, and n holomorphic functions $f_1, \ldots, f_n$ in $\mathbf{A}^{-\infty}(\mathbf{B}_n)$ such that these functions vanish at each ζ_k with multiplicity at least m_k, and each component of the "tube" $S(F;\epsilon,C)$ defined as in (1.2), where $F = (f_1, \ldots, f_n)$, contains at most one point ζ_k and the diameter of such a component is not too "big" in a certain sense (see Theorem 2.7). In the speacial case that $m_k = 1$ for all k, this condition is equivalent to the existence of a lower bound on the Jacobian of F (see Remark 2.8).

2. Prelimilaries and results

First of all, let us fix some notations, which will be used throughout the paper.

Definition 2.1: Let $\mathbf{H}(\mathbf{B}_n)$ be the ring of all holomorphic functions in $\mathbf{B}_n$. Then we define

$$\mathbf{A}^{-\infty}(\mathbf{B}_n) = \left\{ f \in \mathbf{H}(\mathbf{B}_n) : \sup_{z \in \mathbf{B}_n} \frac{\log|f(z)|}{\log \frac{e}{1-|z|}} < \infty \right\}.$$

Note that it is not the specific growth conditions on the functions $\mathbf{A}^{-\infty}(\mathbf{B}_n)$ which are important, but rather their consequences for the ring $\mathbf{A}^{-\infty}(\mathbf{B}_n)$. The growth condition on the holomorphic functions implies that $\mathbf{A}^{-\infty}(\mathbf{B}_n) \supset H^\infty(\mathbf{B}_n)$, the space of bounded holomorphic functions in $\mathbf{B}_n$, and that $\mathbf{A}^{-\infty}(\mathbf{B}_n)$ is closed under differentiation. The main theorem in the paper still holds if the space $\mathbf{A}^{-\infty}(\mathbf{B}_n)$ is replaced by the space

$$A_p(\mathbf{B}_n) := \{f \in \mathbf{H}(\mathbf{B}_n) : |f(z)| \le Ae^{Bp(\frac{1}{1-|z|})}, z \in \mathbf{B}_n, \text{ for some } A, B > 0\},$$

where p is a suitable function so that the calculation in the proof of the paper can be carried out similarily. The space $\mathbf{A}^{-\infty}(\mathbf{B}_n)$ is the union of the weighted spaces

$$A^{-\alpha} := \{f \in \mathbf{H}(\mathbf{B}_n) : \sup_{z \in \mathbf{B}_n} (1-|z|)^\alpha |f(z)| < \infty\}, \quad \alpha > 0,$$

and also the union of the weighted Bergman spaces

$$B_{\alpha,\beta} := \{f \in \mathbf{H}(\mathbf{B}_n) : \int_{\mathbf{B}_n} (1-|z|)^\alpha |f(z)|^\beta dm(z) < \infty\}, \quad \alpha > -1, \beta > 0.$$

It carries the natual topology as an inductive limit of Banach spaces.

Let $f \not\equiv 0$ be a holomorphic function on an open connected neighborhood of $\zeta \in \mathbf{B}_n$. Then a series $f(z) = \sum_{j=\nu}^{\infty} \mathcal{P}_j(z-\zeta)$ converges uniformly on some neighborhood of ζ and represents f on this neighborhood. Here $\mathcal{P}_j$ is a homogeneous polynomial of degree j and $\mathcal{P}_\nu \not\equiv 0$. The nonnegative integer ν, uniquely determined by f and ζ, is called the zero multiplicity, or zero divisor of f at ζ, denoted by $\mathrm{div}_f(\zeta)$.

Let $V = \{(\zeta_k, m_k)\}$ be a multiplicity variety in $\mathbf{B}_n$; that is, a discrete set $\{\zeta_k\} \subset \mathbf{B}_n$ with $|\zeta_k| \to 1$ together with a sequence $\{m_k\}$ of positive integers. Associated to V, there is a unique closed ideal in $\mathbf{H}(\mathbf{B}_n)$,

$$J = J(V) := \{f \in \mathbf{H}(\mathbf{B}_n) : \mathrm{div}_f(\zeta_k) \geq m_k, \forall k\}.$$

Two holomorphic functions g, h in $H(\mathbf{B}_n)$ can be identified modulo J if and only if

$$\frac{\partial^{|I|} g(\zeta_k)}{\partial z^I} = \frac{\partial^{|I|} h(\zeta_k)}{\partial z^I}, \quad 0 \leq |I| < m_k, \quad k \in \mathbf{N}.$$

Here and throughout the paper, we use I to denote a muti-index; that is, $I = (i_1, \ldots, i_n) \in (\mathbf{Z}^+)^n$. The quotient space $\mathbf{H}(\mathbf{B}_n)/J$ can be identified to the space $\mathbf{H}(V)$, the set of all sequences $\{a_{k,I}\}_{k\in\mathbf{N}, 0\leq|I|<m_k}$ of complex numbers, which can be described as "analytic functions " on V. The map

$$\rho : \rho(f) = \left\{\frac{\partial^{|I|} f(\zeta_k)}{I! \partial z^I}\right\}_{k\in\mathbf{N}, 0\leq|I|<m_k}$$

is the natural restriction map from $\mathbf{H}(\mathbf{B}_n)$ into $\mathbf{H}(V)$.

The interpolation problem for $A_p(\mathbf{C}^n)$, the algebra of entire functions satisfying $\sup_{z\in\mathbf{C}^n}\{e^{-Ap(z)}|f(z)|\} < +\infty$ for some $A > 0$, where p is a plurisubharmonic weight, is to study when the map ρ is surjective from $A_p(\mathbf{C}^n)$ to $A_p(V)$, the set of "analytic" functions $\{a_{k,I}\}_{k\in\mathbf{N}, 0\leq|I|<m_k}$ of $H(V)$ satisfying $\sup_{k\in\mathbf{N}}\{e^{-Ap(\zeta_k)} \sum_{|I|=0}^{m_k-1} |a_{k,I}|\} < +\infty$ for some $A > 0$. The corresponding interpolation model for $\mathbf{A}^{-\infty}(\mathbf{B}_n)$ in the unit ball $\mathbf{B}_n$ would be whether ρ is onto from $\mathbf{A}^{-\infty}(\mathbf{B}_n)$ to $A_p(V)$ with $p = \log \frac{1}{1-|z|}$ in our consideration. However, this is not the case. In fact, in the unit ball it is even no longer true that ρ is a map from $\mathbf{A}^{-\infty}(\mathbf{B}_n)$ to $A_p(V)$, as shown by the following

PROPOSITION 2.2: *There exist a multiplicity variety $V = \{(\zeta_k, m_k)\}$ in $\mathbf{B}_n$ and a $f \in \mathbf{A}^{-\infty}(\mathbf{B}_n)$ such that for any $A > 0$,*

$$\sup_{k\in\mathbf{N}}(1-|\zeta_k|)^A\sum_{|I|=0}^{m_k-1}\left|\frac{\partial^{|I|}f(\zeta_k)}{I!\partial^I z}\right| \geq \sup_{k\in\mathbf{N}}(1-|\zeta_k|)^A\sum_{|I|=0}^{m_k-1}\left(\frac{1}{n}(1-|\zeta_k|)\right)^{|I|}\left|\frac{\partial^{|I|}f(\zeta_k)}{I!\partial^I z}\right| = \infty.$$

Thus, $\rho(f) \notin A_p(V)$, where $p = \log\frac{1}{1-|z|}$. However, for any $0 < \lambda < 1/n$ and $A \geq 1$,

$$\sup_{k\in\mathbf{N}}(1-|\zeta_k|)^A\sum_{|I|=0}^{m_k-1}(\lambda(1-|z_k|))^{|I|}\left|\frac{\partial^{|I|}f(\zeta_k)}{I!\partial^I z}\right| < \infty. \tag{2.1}$$

Proof: For the sake of convenience, we look at the case $n = 1$. Consider the function $f(z) = 1/(1-z) \in \mathbf{A}^{-\infty}(\mathbf{B}_n)$. Let $V = \{(\zeta_k, m_k)\}$, where $\zeta_k = 1 - 1/k \in \mathbf{B}_n$ and $m_k = 2^k$, $k = 1, 2, \ldots$. Then $|f^{(j)}(\zeta_k)/j!| = k^{j+1}$. Thus for any $A > 0$,

$$\begin{aligned}\sup_{k\in\mathbf{N}}(1-|\zeta_k|)^A\sum_{|I|=0}^{m_k-1}\left|\frac{\partial^{|I|}f(\zeta_k)}{I!\partial^I z}\right| &\geq \sup_{k\in\mathbf{N}}(1-|\zeta_k|)^A\sum_{j=0}^{m_k-1}\left(\frac{1}{n}(1-|\zeta_k|)\right)^j\left|\frac{f^{(j)}(\zeta_k)}{j!}\right| \\ &= \sup_{k\in\mathbf{N}}\left(\frac{1}{k}\right)^A\sum_{j=0}^{2^k-1}\left(\frac{1}{k}\right)^j k^{j+1} \\ &= \sup_{k\in\mathbf{N}} k^{1-A}\sum_{j=0}^{2^k-1} 1 = \infty.\end{aligned}$$

However, for each $0 < \lambda < 1$ and $A \geq 1$, we have

$$\begin{aligned}\sup_{k\in\mathbf{N}}(1-|\zeta_k|)^A\sum_{j=0}^{m_k-1}(\lambda(1-|\zeta_k|)^j\frac{|f^{(j)}(\zeta_k)|}{j!} &\leq \sup_{k\in\mathbf{N}}\left(\frac{1}{k}\right)^A\sum_{j=0}^{\infty}\left(\frac{\lambda}{k}\right)^j k^{j+1} \\ &= \sup_{k\in\mathbf{N}} k^{1-A}\sum_{j=0}^{\infty}\lambda^j < \infty. \qquad ∎\end{aligned}$$

The above proposition shows that $\rho(\mathbf{A}^{-\infty}(\mathbf{B}_n))$ is not a subspace of $A_p(V)$. However, it leads us to ask if (2.1) is generally true for all $f \in \mathbf{A}^{-\infty}(\mathbf{B}_n)$ and multiplicity varieties V. It is indeed the case, as shown by the following Proposition 2.4, where $l^{-\infty}(V)$ is defined as follows.

Definition 2.3: Let $V = \{(\zeta_k, m_k)\}$ be a multiplicity variety in $\mathbf{B}_n$. Then

$$l^{-\infty}(V) = \{\{a_{k,I}\}_{k\in\mathbf{N},0\le|I|<m_k} : \sup_{k\in\mathbf{N}}(1-|\zeta_k|)^A \sum_{|I|=0}^{m_k-1} |a^*_{k,I}| < \infty \text{ for some } A>0\}$$

where $a^*_{k,I} := (\lambda(1-|\zeta_k|))^{|I|}a_{k,I}$ is the "rectification" of $a_{k,I}$ and $0<\lambda<1/n$ is a fixed constant.

PROPOSITION 2.4: *Let $V = \{(\zeta_k, m_k)\}$ be a multiplicity variety in $\mathbf{B}_n$. Then $\rho(\mathbf{A}^{-\infty}(\mathbf{B}_n)) \subset l^{-\infty}(V)$.*

Proof: Let $f \in \mathbf{A}^{-\infty}(\mathbf{B}_n)$. Then there exist $A, B > 0$ such that $|f(z)| \le A/(1-|z|)^B$. Since $\lambda < 1/n$, there exists a α such that $0 < \lambda n < \alpha < 1$ and so that $\lambda/\alpha < 1/n$. Thus, there exists a $\epsilon > 0$ such that

$$\frac{\lambda}{\alpha} \le \frac{1}{n+\epsilon}. \tag{2.2}$$

Consider

$$g(z) = f(\zeta_k + \alpha(1-|\zeta_k|)z), \quad z\in\mathbf{B}_n.$$

Then we see that

$$|g(z)| \le \frac{A}{[1-(|\zeta_k|+\alpha(1-|\zeta_k|))]^B} = \frac{A}{(1-\alpha)^B(1-|\zeta_k|)^B}. \tag{2.3}$$

Note that g is holomorphic in $\mathbf{B}_n$ and continuous up to the boundary S of $\mathbf{B}_n$. By the Cauchy formula in the uniut ball (see, e.g., [R]) we have

$$g(z) = \int_S \frac{g(w)}{(1-<w,z>)^n} d\sigma(w),$$

where σ is the normalized rotation-invariant positive Borel measure on S and $<w,z>$ is the usual inner product. Thus for $I = (i_1, i_2, \ldots, i_n)$, we have that

$$\frac{\partial^{|I|}g(z)}{\partial^I z} = (-1)^{|I|}n(n+1)\cdots(n+|I|-1)\int_S \frac{w_1^{i_1}\cdots w_n^{i_n}g(w)}{(1-<z,w>)^{n+|I|}}d\sigma(w),$$

where $w = (w_1, \ldots, w_n)$, from which we obtain that

$$\begin{aligned}\left|\frac{\partial^{|I|}g(0)}{\partial^I z}\right| &\le n(n+1)\cdots(n+|I|-1)\int_S |g(w)|d\sigma(w)\\ &\le \frac{An(n+1)\cdots(n+|I|-1)}{(1-\alpha)^B(1-|\zeta_k|)^B}\end{aligned}$$

in view of (2.3) and the fact that $\int_S d\sigma(w) = 1$. But

$$(\alpha(1-|\zeta_k|))^{|I|}\Big|\frac{\partial^{|I|}f(\zeta_k)}{\partial^I z}\Big| = \Big|\frac{\partial^{|I|}g(0)}{\partial^I z}\Big|.$$

We obtain that

$$\begin{aligned}&\sum_{|I|=0}^{\infty}(\lambda(1-|\zeta_k|))^{|I|}\Big|\frac{\partial^{|I|}f(\zeta_k)}{I!\partial^I z}\Big| \\ &\le \frac{A}{(1-\alpha)^B(1-|\zeta_k|)^B}\sum_{|I|=0}^{\infty}\Big(\frac{\lambda}{\alpha}\Big)^{|I|}\frac{n(n+1)\cdots(n+|I|-1)}{I!} \\ &\le \frac{A}{(1-\alpha)^B(1-|\zeta_k|)^B}\sum_{|I|=0}^{\infty}\Big(\frac{1}{n+\epsilon}\Big)^{|I|}\frac{n(n+1)\cdots(n+|I|-1)}{I!}\end{aligned}$$

in view of (2.2). We assert that the series

$$\sum_{|I|=0}^{\infty}\Big(\frac{1}{n+\epsilon}\Big)^{|I|}\frac{n(n+1)\cdots(n+|I|-1)}{I!} < \infty.$$

To see this, consider the holomorphic function

$$h(z) = \frac{1}{\{1-(z_1+z_2+\cdots+z_n)\}^n}$$

in the polydisc $\mathbf{P} := \{z = (z_1, z_2, \ldots, z_n) : |z_1| < 1/n, \ldots, |z_n| < 1/n\}$. The function $h(z)$ can be expanded as the Taylor series in the polydisc $\mathbf{P}$ as follows:

$$h(z) = \sum_{|I|=0}^{\infty}\frac{(-1)^{|I|}n(n+1)\cdots(n+|I|-1)}{I!}z^I.$$

Noting that $z_0 := (-\frac{1}{n+\epsilon}, \ldots, -\frac{1}{n+\epsilon}) \in \mathbf{P}$, we obtain

$$\sum_{|I|=0}^{\infty}\Big(\frac{1}{n+\epsilon}\Big)^{|I|}\frac{n(n+1)\cdots(n+|I|-1)}{I!} = h(z_0) < \infty.$$

Thus, we have

$$\sum_{|I|=0}^{\infty}(\lambda(1-|\zeta_k|))^{|I|}\Big|\frac{\partial^{|I|}f(\zeta_k)}{I!\partial^I z}\Big| \le \frac{A}{(1-\alpha)^B(1-|\zeta_k|)^B}h(z_0) < \infty$$

for some $A, B > 0$. This implies that $\rho(f) \in l^\infty(V)$. The proof is complete. ∎

Remark 2.5: In the above proposition we showed that $\rho(\mathbf{A}^{-\infty}(\mathbf{B}_n)) \subset l^{-\infty}(V)$, i.e., for any $f \in \mathbf{A}^{-\infty}(\mathbf{B}_n)$,

$$\sup_{k\in\mathbf{N}}(1-|\zeta_k|)^A \sum_{|I|=0}^{m_k-1}(\lambda(1-|\zeta_k|))^{|I|}\left|\frac{\partial^{|I|}f(\zeta_k)}{I!\partial^I z}\right| < \infty$$

for some $A > 0$. This is a "precise" result for the unit ball $\mathbf{B}_n$ in the sense that the "rectification factor" $(\lambda(1-|z_k|))^{|I|}$ in the above sum (cf. Definition 2.3) cannot be dropped and $0 < \lambda < 1/n$ is best possible, as shown by Proposition 2.2. In general, the space $l^{-\infty}(V)$ is too large. The interpolation problem with multiplicity we consider is to determine when ρ is surjective from $\mathbf{A}^{-\infty}(\mathbf{B}_n)$ to $l^{-\infty}(V)$. That ρ is surjective means that for any multi-indexed sequence $\{a_{k,I}\} \in l^{-\infty}(V)$ there exists a holomorphic function in $\mathbf{A}^{-\infty}(\mathbf{B}_n)$ such that $f_{k,I} = a_{k,I}$ for any $k \in \mathbf{N}$ and $0 \le |I| < m_k$; i.e., f has a described finite collection of Taylor coefficients. When $m_k = 1$ for all k, then the equality $f_{k,I} = a_{k,I}$ simply means that $f(\zeta_k) = a_k$, where $\{a_k\}$ is a sequence satisfying $\sup_{k\in\mathbf{N}}\{(1-|\zeta_k|)^A|a_k|\} < \infty$ for some constant $A > 0$, which gives the usual notion of interpolating discrete set $\{\zeta_k\}$ for $\mathbf{A}^{-\infty}(\mathbf{B}_n)$, corresponding to the one in $\mathbf{C}^n$ considered in the references mentioned in §1.

Definition 2.6: A multiplicity variety $V = \{(\zeta_k, m_k)\}$ is an interpolating variety for $\mathbf{A}^{-\infty}(\mathbf{B}_n)$ if the restriction map ρ is surjective from $\mathbf{A}^{-\infty}(\mathbf{B}_n)$ to $l^{-\infty}(V)$.

Let $V = \{(\zeta_k, m_k)\}$ be a multiplicity variety. We use $V \subset F^{-1}(0)$, where $F = (f_1, f_2, \dots, f_m)$, to denote that each f_j vanishes at ζ_k with multiplicity at least m_k; i.e., $\operatorname{div}_{f_j}(\zeta_k) \ge m_k, \forall k$. Given $\epsilon, C > 0$, we define $S(F;\epsilon,C)$ by (1.2), which can be regarded as a "tube" of the variety V. Throughout the paper, λ is the fixed number given in Definition 2.3.

We shall prove the following theorem:

THEOREM 2.7: *Let $V = \{(\zeta_k, m_k)\}$ be a multiplicity variety in $\mathbf{B}_n$ and $m \ge n$ a positive integer. Then V is an interpolating variety for $\mathbf{A}^{-\infty}(\mathbf{B}_n)$ if and only if there exist m functions $f_1, f_2, \dots, f_m$ in $\mathbf{A}^{-\infty}(\mathbf{B}_n)$ and two constants $\epsilon, C > 0$ such that $V \subset F^{-1}(0)$, where $F = (f_1, f_2, \dots, f_m)$, and each connected component of $S(F;\epsilon,C) := \{z \in \mathbf{B}_n : |F(z)| < \epsilon(1-|z|)^C\}$ contains at most one point in V and the component containing ζ_k is of dimater at most $\lambda(1-|\zeta_k|)$.*

Remark 2.8: In the case that $m_k = 1$ for all k, one can show that the interpolation condition in Theorem 2.7 is equivalent to the existence of $f_1, f_2, \dots, f_n \in \mathbf{A}^{-\infty}(\mathbf{B}_n)$ such that $V \subset F^{-1}(0)$ and $|JF(\zeta_k)| \ge \epsilon(1-|\zeta_k|)^C, k \in \mathbf{N}$, for some

$\epsilon, C > 0$, where JF is the Jacobian of $F = (f_1, f_2, \dots, f_n)$ (see [BL1] and [M]). We omit the verification.

3. Some lemmas

In the following, we shall use A, B, C, ϵ to denote positive constants, the actual values of which may vary from one occurrence to the next. The number λ is the fixed constant given in Definition 2.3.

To prove the results, we need the following lemmas.

LEMMA 3.1: *Let $V = \{(\zeta_k, m_k)\}$ be an interpolating variety for $\mathbf{A}^{-\infty}(\mathbf{B}_n)$. Then given $M > 0$ there exist two constants $l > 0$ and $\epsilon > 0$ such that*

$$A_l(V) \supset \{a = \{a_{k,I}\}_{k\in\mathbf{N},|I|<m_k} : ||a|| < \epsilon\},$$

where

$$||a|| = \sup_{k\in\mathbf{N}} (1 - |\zeta_k|)^M \{ \sum_{|I|=0}^{m_k-1} |a^*_{k,I}| \},$$

$$a^*_{k,I} = (\lambda(1 - |\zeta_k|)^I a_{k,I},$$

$$A_l(V) = \{a_f := \{f_{k,I}\}_{k\in\mathbf{N},|I|<m_k} : f \in A_l(\mathbf{B}_n), ||a_f|| \le 1\},$$

and

$$A_l(\mathbf{B}_n) = \{f \in \mathbf{A}^{-\infty}(\mathbf{B}_n) : (1 - |z|)^l |f(z)| \le l, z \in \mathbf{B}_n\}.$$

Proof: Let

$$\mathcal{A} = \{a = \{a_{k,I}\}_{k\in\mathbf{N},|I|<m_k} : ||a|| \le 1\}.$$

Then it is easy to check that $\mathcal{A}$ is complete under the metric induced by the norm $||a||$. Because V is an interpolating variety for $\mathbf{A}^{-\infty}(\mathbf{B}_n)$, for any sequence $a = \{a_{k,I}\} \in \mathcal{A}$, there exists a $f \in A_l(\mathbf{B}_n)$ for some l such that $f_{k,I} = a_{k,I}$ for $k \in \mathbf{N}$ and $|I| < m_k$. That is, $a \in A_l(V)$. This shows that $\mathcal{A} = \bigcup_{l=1}^{\infty} A_l(V)$.

One can check that each $A_l(V)$ is a closed subset of $\mathcal{A}$. In fact, if $\{f_j\}$ is a sequence in $A_l(\mathbf{B}_n)$ such that $(f_j)_{k,I} \to a \in \mathcal{A}$ as $j \to \infty$, then by the definition of $\mathbf{A}^{-\infty}(\mathbf{B}_n)$, $\{f_j\}$ is uniformly bounded on each closed subset of $\mathbf{B}_n$. Using Montel's theorem (see, e.g., [G]) we know that $\{f_j\}$ is a normal family in $\mathbf{B}_n$. By passing to a subsequence, we can assume that $f_j \to f$ normally, where f is the limit function. By the Weierstrass theorem and the uniqueness of the limit (see, e.g., [G]), we deduce that $f \in A_l(\mathbf{B}_n)$ and $\{f_{k,I}\} = a$. It follows that $a \in A_l(V)$ and thus that $A_l(V)$ is closed. Now by the well-known Baire-category theorem, we know that for some l, $A_l(V)$ has a non-empty interior. Therefore, there exists a $\epsilon > 0$ such that $A_l(V) \supset \{a = \{a_{k,I}\}_{k\in\mathbf{N},|I|<m_k} : ||a|| < \epsilon\}$. ∎

LEMMA 3.2: *Let $V = \{(\zeta_k, m_k)\}$ be an interpolating variety for $A_p(\mathbf{B}_n)$. Then there exist two constants $l > 0$ and $\epsilon > 0$ such that the following two conclusions hold:*

(i) *There exists a sequence $\{f_k\}$ of holomorphic functions in $\mathbf{B}_n$ such that*

$$(1-|z|)^l |f_k(z)| \le l, \quad z \in \mathbf{B}_n \quad \text{and} \quad k \in \mathbf{N} \tag{3.1}$$

and $(f_k)_{j,I} = 0$ for all j and $|I| \le m_j - 1$ except that

$$\frac{\partial^{m_k-1} f_k(\zeta_k)}{(m_k-1)!\partial z_1^{m_k-1}} = \frac{\epsilon}{(\lambda(1-|\zeta_k|))^{m_k-1}}. \tag{3.2}$$

(ii) *There exists a sequence $\{g_k\}$ of holomorphic functions in $\mathbf{B}_n$ such that each g_k satisfies (3.1) and $(g_k)_{j,I} = 0$, $\forall j$, $|I| \le m_j - 1$ except that*

$$g_k(\zeta_k) = \epsilon. \tag{3.3}$$

Proof: It follows from Lemma 3.1 that there exist two constants $l > 0$ and $\epsilon > 0$ such that the space $A_l(V)$ contains the space $\mathcal{S} := \{a = \{a_{k,I}\}_{k\in\mathbf{N},|I|<m_k} : ||a|| < \epsilon\}$, where $A_l(V)$ is the same as in Lemma 3.1. Consider the sequence $a_k = \{a_{k,I}\}_{k\in\mathbf{N},|I|<m_k}$ satisfying $a_{j,I} = 0$ for all j and $0 \le |I| < m_j$ except that

$$a_{k,I_k} = \frac{\epsilon}{(\lambda(1-|\zeta_k|))^{m_k-1}},$$

where $I_k = (m_k - 1, 0, \ldots, 0) \in (\mathbf{Z}^+)^n$. Then it is clear that $a_k \in \mathcal{S}$. Hence by Lemma 3.1 there exist holomorphic functions f_k in $\mathbf{B}_n$ such that (3.1) holds and that $(f_k)_{j,I} = a_{j,I}$ for all j and $0 \le |I| < m_j$. Thus (3.2) holds. The conclusion (ii) follows from the same argument. ∎

LEMMA 3.3 (Schwarz, [G]): *If f is holomorphic in an open neighborhood of a closed ball $\bar{B}(\zeta, r)$ in $\mathbf{C}^n$ centered at ζ and with radius r, $|f(z)| \le M$ for $z \in B(\zeta, r)$, and $\frac{\partial^{|I|} f}{\partial z^I}(\zeta) = 0$ whenever $|I| < m$ for some $m \in \mathbf{N}$, then $|f(z)| \le M r^{-m} |z-\zeta|^m$ for $z \in \bar{B}(\zeta, r)$.*

LEMMA 3.4: *Let $V = \{(\zeta_k, m_k)\}$ be an interpolating variety for $\mathbf{A}^{-\infty}(\mathbf{B}_n)$. Then*

(i) *$\lambda^{m_k} \ge \epsilon(1-|\zeta_k|)^C$ for some $\epsilon, C > 0$, where $0 < \lambda < 1/n$ is the fixed constant in Definition 2.3;*

(ii) *$\sum_{k=1}^{\infty}(1-|\zeta_k|)^M < \infty$ for large $M > 0$.*

Proof: By Lemma 3.2 (i) there exists a sequence of funcctions $\{f_k\}$ satisfying (3.1) and (3.2). By the Cauchy Theorem,

$$\frac{1}{(m_k-1)!}\frac{\partial^{m_k-1} f_k(\zeta_k)}{\partial z_1^{m_k-1}} = \left(\frac{1}{2\pi i}\right)^n \int \frac{f_k(z) dz_1 \wedge dz_2 \cdots \wedge dz_n}{(z_1-\zeta_{1,k})^{m_k}(z_2-\zeta_{2,k})\cdots(z_n-\zeta_{n,k})}, \tag{3.4}$$

where $\zeta_k = (\zeta_{1,k}, \zeta_{2,k}, \ldots, \zeta_{n,k})$, and the integral is taken over the boundary of the polydisc

$$P := \{z : |z_1 - \zeta_{1,k}| < \lambda^{\frac{1}{2}}(1 - |\zeta_k|), \ldots, |z_n - \zeta_{n,k}| < \lambda^{\frac{1}{2}}(1 - |\zeta_k|)\} \\ \subset \{z : |z - \zeta_k| < (n\lambda)^{\frac{1}{2}}(1 - |\zeta_k|)\}.$$

By (3.1), for $z \in P$,

$$|f_k(z)| \leq \frac{l}{(1 - |z|)^l} \leq \frac{A}{(1 - |\zeta_k|)^B}$$

for some $A, B > 0$ since

$$1 - |z| \geq 1 - (|z - \zeta_k| + |\zeta_k|) \geq (1 - (n\lambda)^{\frac{1}{2}})(1 - |\zeta_k|).$$

Therefore we have, in view of (3.2) and (3.4), that

$$\frac{\epsilon}{(\lambda(1 - |\zeta_k|))^{m_k - 1}} \leq \frac{A}{(1 - |\zeta_k|)^B} \frac{1}{(\lambda^{\frac{1}{2}}(1 - |\zeta_k|))^{m_k - 1}}$$

and so that $\lambda^{m_k} \geq \epsilon(1 - |\zeta_k|)^C$ for some $\epsilon, C > 0$. That is, (i) holds.

For each k, there exists a j such that $d_k = |\zeta_k - \zeta_j|$, where

$$d_k := \inf_{j \neq k}\{|\zeta_j - \zeta_k|\}.$$

Let g_k be the functions in Lemma 3.2 satisfying (3.1) and (3.3). Set $h_k(z) = g_k(z) - g_k(\zeta_k)$. Then $h_k(\zeta_k) = 0$. When $z \in U := \{z : |z - \zeta_k| < \lambda(1 - |\zeta_k|)\}$ we have

$$1 - |z| \geq 1 - (|z - \zeta_k| + |\zeta_k|) \geq (1 - \lambda)(1 - |\zeta_k|)$$

and so

$$|h_k(z)| \leq \frac{l}{(1 - |z|)^l} + \epsilon \leq \frac{A}{(1 - |\zeta_k|)^B}.$$

It follows from Lemma 3.3 that

$$|h_k(z)| \leq \frac{A|z - \zeta_k|}{(1 - |\zeta_k|)^B}.$$

Hence, if $\zeta_j \in U$ with $g_k(\zeta_j) = 0$ we will have, in view of (3.3),

$$\epsilon = |g_k(\zeta_k)| = |h_k(\zeta_j)| \leq \frac{A|\zeta_k - \zeta_j|}{(1 - |\zeta_k|)^B}$$

and so $|\zeta_j - \zeta_k| \geq \epsilon(1-|\zeta_k|)^C$ for some $\epsilon, C > 0$. If $\zeta_j \notin U$, then $|\zeta_j - \zeta_k| \geq \lambda(1-|\zeta_k|)$ by the definition of the set U. Therefore, in any case we always have $d_k \geq \epsilon(1-|\zeta_k|)^C$ for some $0 < \epsilon < 1$ and $C > 0$. One can choose $0 < \epsilon < 1$ and $C > 1$ such that the ball B_k centered at ζ_k with radius $\epsilon(1-|\zeta_k|)^C \leq d_k$ is contained in the ball $\mathbf{B}_n$. Then $B_k \cap B_j = \emptyset$ for $i \neq j$ and the volume $|B_k|$ of B_k satisfies $|B_k| \geq \epsilon(1-|\zeta_k|)^C$ for some $\epsilon, C > 0$. When $z \in B_k$,

$$|1-z| \geq |1-\zeta_k| - |z-\zeta_k| \geq 1 - |\zeta_k| - \epsilon(1-|\zeta_k|)^C,$$

which implies that $1-|\zeta_k| \leq (1-|z|)/(1-\epsilon)$. Thus, we deduce that

$$\begin{aligned}\sum_{k=1}^{\infty}(1-|\zeta_k|)^M &= \sum_{k=1}^{\infty}\frac{1}{|B_k|}\int_{B_k}(1-|\zeta_k|)^M dm \\ &\leq A\sum_{k=1}^{\infty}\int_{B_k}(1-|\zeta_k|)^{M-C}dm \\ &\leq A\sum_{k=1}^{\infty}\int_{B_k}(1-|z|)^{M-C}dm \\ &\leq A\int_{\mathbf{B}_n}(1-|z|)^{M-C}dm < \infty\end{aligned}$$

for large M. This completes the proof. ∎

4. Proof of Theorem 2.7

We first prove the necessity. Given a $M > 0$, by Lemma 3.1, there exists a positive integer l and a ϵ_0 such that

$$A_l(V) \supset \{\{a_{k,I}\}_{k\in\mathbf{N},|I|<m_k} : \sup_{k\in\mathbf{N}}\{(1-|\zeta_k|)^M \sum_{|I|=0}^{m_k-1} |a^*_{k,I}|\} \leq \epsilon_0\}.$$

Here we use the same notations as in Lemma 3.1. Thus, for each $1 \leq j \leq n$ we can obtain sequences of holomorphic functions $\{g_{j,k}\}$ with $g_{j,k} \in A_l(\mathbf{B}_n)$ for any $k \in \mathbf{N}$ and $1 \leq j \leq n$ such that $(g_{j,k})_{i,I} = 0$, $\forall i$, $|I| < m_i$ except that

$$\frac{\partial^{l_k} g_{j,k}}{\partial z_j^{l_k}}(\zeta_k) = \frac{\epsilon_0}{(\lambda(1-|\zeta_k|))^{l_k}(1-|\zeta_k|)^M}, \tag{4.1}$$

where $l_k = m_k/2$ if m_k is even and $l_k = (m_k-1)/2$ if m_k is odd. We define, for $1 \leq j \leq n$, the following functions:

$$f_j(z) = \sum_{k=1}^{\infty} h_{j,k}(1-|\zeta_k|)^{2M}, \tag{4.2}$$

where $h_{j,k} = g_{j,k}^2(z)$ if m_k is even and $h_{j,k} = (z_j - \zeta_{j,k})g_{j,k}^2(z)/\lambda(1-|\zeta_k|)$ if m_k is odd, $z = (z_1, \dots, z_n)$ and $\zeta_k = (\zeta_{1,k}, \dots, \zeta_{n,k})$. It is clear that $\mathrm{div}_{f_j}(\zeta_k) \geq m_k$ and so $V \subset F^{-1}(0)$, where $F = (f_1, \dots, f_n)$. We claim that $f_j \in \mathbf{A}^{-\infty}(\mathbf{B}_n)$ for each $1 \leq j \leq n$. In fact, since $g_{j,k} \in A_l(\mathbf{B}_n)$ for $k \in \mathbf{N}$, we have that for $z \in \mathbf{B}_n$, $|g_{j,k}(z)| \leq l/(1-|z|)^l$. Therefore, we deduce that

$$|g_{j,k}^2(z)|(1-|\zeta_k|)^{2M} \leq \frac{l^2(1-|\zeta_k|)^{2M}}{(1-|z|)^{2l}}.$$

By Lemma 3.4 (ii), taking M sufficiently large, we see that the series (4.2) is uniformly convergent in closed sets of $\mathbf{B}_n$, and moreover $|f_j(z)| \leq A/(1-|z|)^{2l}$, $z \in \mathbf{B}_n$, for some constants $A > 0$; that is, $f_j \in \mathbf{A}^{-\infty}(\mathbf{B}_n)$.

Next we show that there are positive constants ϵ, C such that the "tube" $S(F; \epsilon, C)$ satisfies the conclusion of the theorem. To this end, let $k > 0$ and let $u = (u_1, \dots, u_n)$ be a unit vector in $\mathbf{C}^n$. Then there exists a j $(1 \leq j \leq n)$ such that $u_j \geq 1/\sqrt{n}$. For this fixed j, consider the Taylor expansion of $f_j(z)$ at ζ_k. One can verify that, in view of (4.1) and (4.2),

$$f_j(z) = \epsilon_0^2(\lambda(1-|\zeta_k|))^{-m_k}(z_j - \zeta_{j,k})^{m_k} + \sum_{|I| \geq m_k + n_k}^{\infty} C_I(z-\zeta_k)^I,$$

where $n_k = m_k/2$ if m_k is even and $n_k = (m_k+1)/2$ if m_k is odd; C_I's are complex numbers. From the above expansion, we deduce that for $w \in \mathbf{B}_1 \subset \mathbf{C}$,

$$F_j(w) := f_j(\zeta_k + u\sqrt{\lambda}(1-|\zeta_k|)w) = \epsilon_1 w^{m_k} + \sum_{j \geq m_k + n_k} b_j w^j, \tag{4.3}$$

where $\epsilon_1 = \epsilon_0^2 u_j^{m_k} \sqrt{\lambda}^{-m_k}$ and b_j's are complex numbers. Noting that $\lambda < 1/n$, we obtain that

$$\epsilon_1 \geq \epsilon_0^2(1/\sqrt{n})^{m_k}(1/\sqrt{n})^{-m_k} = \epsilon_0^2. \tag{4.4}$$

Let

$$d_u = \min\{1, \mathrm{dist}(0, F_j^{-1}(0) \setminus \{0\})\}$$

and set $G_j(w) = F_j(w)/w^{m_k}$. Then $|G_j(w)| \leq A/(1-|\zeta_k|)^B$ for some constants $A, B > 0$ on $|w| = 1$ and thus in $|w| \leq 1$ by the maximum modulus theorem. Also let $H_j(w) = G_j(w) - G_j(0)$. Then by (4.3), we see that $H_j(w)$ has a zero at $w = 0$ of order at least n_k. Note that $|H_j(w)| \leq A/(1-|\zeta_k|)^B$ for some constants $A, B > 0$ on $|w| \leq 1$. We have, by Lemma 3.3, that

$$|H_j(w)| \leq \frac{A}{(1-|\zeta_k|)^B}|w|^{n_k}$$

on $|w| \leq 1$. Thus, if $a \neq 0$ is a zero of $F_j(w)$ in $|w| < 1$, then $G_j(a) = 0$ and thus

$$|H_j(a)| = |G_j(0)| = \epsilon_1 \geq \epsilon_0^2$$

by (4.3) and (4.4), from which it follows that

$$|a|^{n_k} \geq |H_j(a)| A^{-1}(1 - |\zeta_k|)^B$$

and thus that $d_u^{n_k} > \epsilon(1 - |\zeta_k|)^C$ for some constants $\epsilon, C > 0$, which implies

$$d_u^{m_k} > \epsilon(1 - |\zeta_k|)^C.$$

Therefore,

$$d_u > \epsilon^{1/m_k}(1 - |\zeta_k|)^{C/m_k} := \frac{1}{\lambda} d_k, \tag{4.5}$$

where $0 < \epsilon < 1$ and C are two constants. Note that $G_j(w)$ has no zero in $|w| \leq \frac{1}{\lambda} d_k$ by the construction of d_u. Recall the following result from the Carathéodory theorem (see, e.g., [L]): If h is holomorphic and has no zero in $|w| \leq R$ with $h(0) = 1$, then

$$\log|h(w)| \geq -\frac{2r}{R - r} \log \max_{|w|=R} \{|h(w)|\} \quad \text{for } |w| \leq r < R.$$

Applying it to $G_j(w)$ in $|w| \leq \frac{1}{\lambda} d_k$ we deduce that for $|w| \leq d_k < \frac{1}{\lambda} d_k$,

$$\log\left|\frac{G_j(w)}{G_j(0)}\right| \geq -\frac{2\lambda}{1 - \lambda} \log\left(\max_{|w|=d_k/\lambda} \left\{\left|\frac{G_j(w)}{G_j(0)}\right|\right\}\right),$$

which implies that $|G_j(w)| \geq \epsilon(1 - |\zeta_k|)^C$ for some constants $\epsilon, C > 0$. In particular, for $|w| = d_k$,

$$\begin{aligned} |F_j(w)| &= |w^{m_k} G_j(w)| \\ &\geq \{\lambda \epsilon^{1/m_k}(1 - |\zeta_k|)^{C/m_k}\}^{m_k} \epsilon(1 - |\zeta_k|)^C \geq \epsilon(1 - |\zeta_k|)^C \end{aligned}$$

for some $\epsilon, C > 0$ by virtue of (4.5) and Lemma 3.4(i).

So far we have proved that for a given unit vector $u \in \mathbf{C}^n$, there exists a j $(1 \leq j \leq n)$ such that $|f_j(\zeta_k + u\sqrt{\lambda}(1 - |\zeta_k|)w)| \geq \epsilon(1 - |\zeta_k|)^C$ on $|w| = d_k$, where the constants ϵ, C are independent of u and k. Therefore, for $z \in \mathbf{B}_n$ with $|z - \zeta_k| = \sqrt{\lambda}(1 - |\zeta_k|) d_k$, we always have

$$|F(z)| = \left(\sum_{j=1}^{n} |f_j(z)|^2\right)^{\frac{1}{2}} \geq \epsilon(1 - |\zeta_k|)^C \geq \epsilon(1 - |z|)^C$$

for some $\epsilon, C > 0$. Now consider the neighborhood

$$U_k := \{z \in \mathbf{B}_n : |z - \zeta_k| \leq \sqrt{\lambda}(1 - |\zeta_k|)d_k\}$$

of ζ_k. By the above result, we know that $|F(z)| \geq \epsilon(1-|z|)^C$ on ∂U_k. Recall that $S(F; \epsilon, C) = \{z \in \mathbf{B}_n : |F(z)| < \epsilon(1 - |z_k|)^C\}$. Thus the connected component V_k of $S(F; \epsilon, C)$ containing ζ_k is clearly contained in U_k. By the construction of d_k, we see that U_k, and thus V_k, has diameter less than $\lambda(1 - |\zeta_k|)$ and does not contain other points of V. Finally, if $m > n$, we can easily add $m - n$ entire functions $f_{n+1}, \dots, f_m$ so that $F = (f_1, f_2, \dots, f_m)$ satisfy the conclusion of the theorem. This completes the proof of the necessity.

To prove the sufficiency, let V_k be the connected component of $S(F; \epsilon, C)$ containing ζ_k. Suppose $\{a_{k,I}\} \subset l^{-\infty}(V)$ is a given multi-indexed sequence with

$$\sum_{|I|=0}^{m_k-1} (\lambda(1 - |\zeta_k|))^I |a_{k,I}| \leq \frac{A}{(1 - |\zeta_k|)^B}$$

for some constants $A, B > 0$. We define an analytic function $\gamma : S(F; \epsilon, C) \to \mathbf{C}$ by

$$\gamma(z) = \begin{cases} \sum_{|I|=0}^{m_k-1} a_{k,I}(z - \zeta_k)^I, & \text{if } z \in V_k; \\ 0, & \text{if } z \in S(F; \epsilon, C) \setminus \bigcup_{k\in\mathbf{N}} V_k. \end{cases}$$

Then it is clear that $\gamma_{k,I} = a_{k,I}$ for $k \in \mathbf{N}$ and $0 \leq |I| \leq m_k - 1$. Since $|z - \zeta_k| \leq \lambda(1 - |\zeta_k|)$ on V_k by the assumption of the theorem, we see that, for $z \in V_k$,

$$|\gamma(z)| \leq \sum_{|I|=0}^{m_k-1} (\lambda(1 - |\zeta_k|))^I |a_{k,I}| \leq \frac{A}{(1 - |\zeta_k|)^B}.$$

Note that for $z \in V_k$,

$$1 - |\zeta_k| \geq 1 - (|\zeta_k - z| + |z|) \geq (1 - |z|) - \lambda(1 - |\zeta_k|)$$

or $1 - |\zeta_k| \geq (1 - |z|)/(1 + \lambda)$. We deduce that

$$|\gamma(z)| \leq \frac{A}{(1 - |z|)^B} \tag{4.6}$$

for some $A, B > 0$ for $z \in V_k$ and thus for all $z \in S(F, \epsilon, C)$, since $\gamma(z) = 0$ for $z \in S(F; \epsilon, C) \setminus \bigcup_{k\in\mathbf{N}} V_k$. We will extend γ to a holomorphic function in $\mathbf{A}^{-\infty}(\mathbf{B}_n)$ by L^2-estimates for $\bar\partial$ equations (cf. [BT], [H] and [KT]). Since $\partial f_j/\partial z_i \in \mathbf{A}^{-\infty}(\mathbf{B}_n)$,

$$\sum_{i=1}^{n} \left|\frac{\partial f_j}{\partial z}(z)\right| \leq \frac{A}{(1 - |z|)^B}$$

for $z \in \mathbf{B}_n$ and some $A, B > 0$. Take a small $\epsilon_1 < \epsilon$ and a large $C_1 > C$. We assert that the distance $d(z)$ of a point $z \in S(F; \epsilon_1, C_1)$ to the complement of $S(F; \epsilon, C)$ satisfies

$$d(z) \geq \epsilon_2 (1 - |z|)^{C_2} \tag{4.7}$$

for some small $\epsilon_2 < \epsilon_1$ and large $C_2 > C_1$. Otherwise there would be a w on the boundary of $S(F; \epsilon, C)$ such that $|w - z| \leq \epsilon_2 (1 - |z|)^{C_2}$. Then for each $1 \leq j \leq n$,

$$\begin{aligned} |f_j(w) - f_j(z)| &= \int_0^1 \frac{df_j}{dt}(z + (w - z)t) dt| \\ &\leq |w - z|| \int_0^1 \sum_{i=1}^n \frac{\partial f_i}{\partial z_i}(z + (w - z)t) dt| \\ &\leq \epsilon_2 (1 - |z|)^{C_2} \frac{A}{(1 - |z|)^B}. \end{aligned}$$

Then

$$\begin{aligned} |f_j(w)| &\leq |f_j(z)| + |f_j(z) - f_j(w)| \\ &\leq \epsilon_1 (1 - |z|)^{C_1} + \epsilon_2 (1 - |z|)^{C_2} \frac{A}{(1 - |z|)^B} \end{aligned}$$

and thus

$$|F(w)| = \left(\sum_{j=1}^n |f_j(w)|^2 \right)^{\frac{1}{2}} \leq \epsilon (1 - |z|)^C$$

if ϵ_1, ϵ_2 are taken sufficiently small and C_1, C_2 sufficiently large. It contradicts the choice of the point of w. Now we can use the Whitney theorem (see, e.g., [BG1, p. 18]) to choose a cut-off function $\chi \in C^\infty$ such that $0 \leq \chi \leq 1$,

$$|\bar{\partial} \chi| \leq A (d(z))^{-1} \leq \frac{A}{(1 - |z|)^B}$$

for some $A, B > 0$, $\chi = 1$ on $S(F; \epsilon_1, C_1)$ and $\chi = 0$ on a neighborhood of the complement of $S(F; \epsilon, C)$. Note that $\phi := \gamma \bar{\partial} \chi$ is a $\bar{\partial}$ closed form. By virtue of (4.6) and the fact that $|F(z)| \geq \epsilon_1 (1 - |z|)^{C_1}$ for $z \in \operatorname{supp}(\phi)$, we see that for each $\alpha > 0$ there exists a $\beta > 0$ such that

$$\int_{\mathbf{B}_n} \frac{|\phi(z)|^2}{|F(z)|^\alpha} (1 - |z|)^\beta dm(z) < \infty.$$

By Theorem 2.6 in [KT] there exist $\bar{\partial}$ closed $(0,1)$-forms $\phi_1, \phi_2, \ldots, \phi_m$ and some $q > 0$ such that $\phi = \phi_1 f_1 + \cdots + \phi_m f_m$ and

$$\int_{\mathbf{B}_n} |\phi_j(z)|^2 (1 - |z|)^q dm(z) < \infty.$$

Thus by Hörmander's theorem [H], there exist solutions ψ_j to the $\bar{\partial}$ equations $\bar{\partial}\psi_j = \phi_j$ satisfying the L^2-estimate:

$$\int_{\mathbf{B}_n} |\psi_j(z)|^2(1-|z|)^q dm(z) < \infty.$$

Define $f = \gamma\chi - \sum_{j=1}^m \psi_j f_j$. Then

$$\bar{\partial} f = \gamma\bar{\partial}\chi - \sum_{j=1}^m f_j \bar{\partial}\psi_j = \phi - \sum_{j=1}^m f_j\phi_j = 0$$

and furthermore

$$\int_{\mathbf{B}_n} |f(z)|^2(1-|z|)^A dm(z) < \infty,$$

for some $A > 0$, which implies that f is in a weighted Bergman space and thus in $\mathbf{A}^{-\infty}(\mathbf{B}_n)$. By checking the Taylor expansion of f about ζ_k, we easily see that $f_{k,I} = \gamma_{k,I} = a_{k,I}$ for $k \in \mathbf{N}$ and $0 \le |I| \le m_k - 1$. This shows that V is an interpolating variety for $A_p(\mathbf{C}^n)$. ∎

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